Exercise 16

Evaluate the line integral, where C is the given curve.

$$\int_C (y+z)\,dx + (x+z)\,dy + (x+y)\,dz, \quad C \text{ consists of line segments from } (0,0,0) \text{ to } (1,0,1) \text{ and from } (1,0,1) \text{ to } (0,1,2)$$

Solution

The equation of the line going from $(0,0,0)$ to $(1,0,1)$ is

$$\begin{aligned}\mathbf{y} &= \mathbf{m}t + \mathbf{b} \\ &= \langle 1-0, 0-0, 1-0\rangle t + \langle 0,0,0\rangle \\ &= \langle t, 0, t\rangle,\end{aligned}$$

and the equation of the line going from $(1,0,1)$ to $(0,1,2)$ is

$$\begin{aligned}\mathbf{y} &= \mathbf{m}t + \mathbf{b} \\ &= \langle 0-1, 1-0, 2-1\rangle t + \langle 1,0,1\rangle \\ &= \langle -t, t, t\rangle + \langle 1,0,1\rangle \\ &= \langle -t+1, t, t+1\rangle,\end{aligned}$$

where $0 \le t \le 1$. Write the integral in terms of a dot product.

$$\int_C (y+z)\,dx + (x+z)\,dy + (x+y)\,dz = \int_C \langle y+z, x+z, x+y\rangle \cdot \langle dx, dy, dz\rangle$$

Split it up over the two lines.

$$\int_C (y+z)\,dx+(x+z)\,dy+(x+y)\,dz = \int_{\text{Line 1}} \langle y+z, x+z, x+y\rangle\cdot\langle dx, dy, dz\rangle + \int_{\text{Line 2}} \langle y+z, x+z, x+y\rangle\cdot\langle dx, dy, dz\rangle$$

With the parameterization, $x(t) = t$ and $y = 0$ and $z = t$, for the first line and the parameterization, $x = -t+1$ and $y = t$ and $z = t+1$, for the second line, the line integral becomes

$$\begin{aligned}\int_C (y+z)\,dx + (x+z)\,dy + (x+y)\,dz &= \int_0^1 \langle y(t)+z(t), x(t)+z(t), x(t)+y(t)\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle dt \\ &\quad + \int_0^1 \langle y(t)+z(t), x(t)+z(t), x(t)+y(t)\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle dt \\ &= \int_0^1 \langle 0+t, t+t, t+0\rangle \cdot \langle 1,0,1\rangle\,dt \\ &\quad + \int_0^1 \langle t+(t+1), (-t+1)+(t+1), (-t+1)+t\rangle \cdot \langle -1,1,1\rangle\,dt.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_C (y+z)\,dx + (x+z)\,dy + (x+y)\,dz &= \int_0^1 \langle t, 2t, t\rangle \cdot \langle 1, 0, 1\rangle + \int_0^1 \langle 2t+1, 2, 1\rangle \cdot \langle -1, 1, 1\rangle\,dt \\ &= \int_0^1 [t(1) + 2t(0) + t(1)]\,dt + \int_0^1 [(2t+1)(-1) + 2(1) + 1(1)]\,dt \\ &= \int_0^1 (2t)\,dt + \int_0^1 (-2t+2)\,dt \\ &= \cancel{\int_0^1 (2t)\,dt} - \cancel{\int_0^1 (2t)\,dt} + 2\int_0^1 dt \\ &= 2.\end{aligned}$$